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Some properties of certain analytic functions

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Abstract

Defining the subclasses $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ of certain analytic functions $f(z)$ in the open unit disk \mathbb{U} , some properties for $f(z)$ belonging to the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ are discussed. In this present paper, some coefficient estimates and some interesting applications of Jack's lemma for functions $f(z)$ in the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ are given.

1 Introduction

Let \mathcal{A} be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. Shams, Kulkarni and Jahangiri [3] have considered the subclass $\mathcal{SD}(\alpha, \beta)$ of \mathcal{A} consisting of $f(z)$ which satisfy

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

for some $\alpha (\alpha \geq 0)$ and $\beta (0 \leq \beta < 1)$. The class $\mathcal{KD}(\alpha, \beta)$ is defined by the subclass of \mathcal{A} consisting of $f(z)$ such that $zf'(z) \in \mathcal{SD}(\alpha, \beta)$. In view of the classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$, we introduce the subclass $\mathcal{MD}(\alpha, \beta)$ of \mathcal{A} consisting of all functions $f(z)$ which satisfy

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

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for some $\alpha (\alpha \leq 0)$ and $\beta (\beta > 1)$. The class $\mathcal{ND}(\alpha, \beta)$ is also defined by $f(z) \in \mathcal{ND}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha, \beta)$. The classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ were introduced by Nishiwaki and Owa [2]. We discuss some properties of functions $f(z)$ belonging to the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$.

We note if $f(z) \in \mathcal{MD}(\alpha, \beta)$, then $\frac{zf'(z)}{f(z)} = u + iv$ maps \mathbb{U} onto elliptic domain such that

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 + \frac{\alpha^2}{\alpha^2 - 1}v^2 < \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

for $\alpha < -1$, the parabolic domain such that

$$u < -\frac{1}{2(\beta - 1)}v^2 + \frac{\beta + 1}{2}$$

for $\alpha = -1$, and the hyperbolic domain such that

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 - \frac{\alpha^2}{1 - \alpha^2}v^2 > \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

for $-1 < \alpha < 0$.

2 Coefficient estimates for the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$

By definitions of $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$, we derive

Theorem 2.1. *If $f(z) \in \mathcal{MD}(\alpha, \beta)$, then*

$$f(z) \in \mathcal{MD}\left(0, \frac{\beta - \alpha}{1 - \alpha}\right).$$

Proof. If $f(z) \in \mathcal{MD}(\alpha, \beta)$,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \leq \alpha \operatorname{Re}\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta \quad (z \in \mathbb{U})$$

implies that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{\beta - \alpha}{1 - \alpha} \quad (\alpha \leq 0, \beta > 1).$$

Since $\frac{\beta - \alpha}{1 - \alpha} > 1$, we prove the Theorem. □

Corollary 2.1. *If $f(z) \in \mathcal{ND}(\alpha, \beta)$, then*

$$f(z) \in \mathcal{ND}\left(0, \frac{\beta - \alpha}{1 - \alpha}\right).$$

Our result for the coefficient estimates of $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ is contained in

Theorem 2.2. *If $f(z) \in \mathcal{MD}(\alpha, \beta)$, then*

$$|a_2| \leq \frac{2(\beta - 1)}{1 - \alpha}$$

and

$$|a_n| \leq \frac{2(\beta - 1)}{(n - 1)(1 - \alpha)} \prod_{j=1}^{n-2} \left(1 + \frac{2(\beta - 1)}{j(1 - \alpha)}\right) \quad (n \geq 3).$$

Proof. If $f(z) \in \mathcal{MD}(\alpha, \beta)$, then

$$\beta - \alpha + (\alpha - 1) \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0$$

from Theorem 2.1. And let us define the function $p(z)$ by

$$(2.1) \quad p(z) = \frac{\beta - \alpha + (\alpha - 1) \frac{zf'(z)}{f(z)}}{\beta - 1}.$$

Then $p(z)$ is analytic in \mathbb{U} , $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$). Therefore, if we write

$$(2.2) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then $|p_n| \leq 2$ ($n \geq 1$). From (2.1) and (2.2), we obtain that

$$(\alpha - 1) \sum_{n=2}^{\infty} (n - 1) a_n z^n = (\beta - 1) \sum_{n=1}^{\infty} p_n z^n \left(z + \sum_{n=2}^{\infty} a_n z^n \right).$$

Therefore we have

$$a_n = \frac{\beta - 1}{(n - 1)(\alpha - 1)} (p_{n-1} + p_{n-2} a_2 + \cdots + p_2 a_{n-2} + p_1 a_{n-1})$$

for all $n \geq 2$. When $n = 2$,

$$|a_2| \leq \frac{\beta - 1}{1 - \alpha} |p_1| \leq \frac{2(\beta - 1)}{1 - \alpha}.$$

And when $n = 3$,

$$\begin{aligned} |a_3| &\leq \frac{\beta-1}{2(1-\alpha)}(|p_2| + |p_1||a_2|) \\ &\leq \frac{2(\beta-1)}{2(1-\alpha)} \left(1 + \frac{2(\beta-1)}{1-\alpha}\right). \end{aligned}$$

Let us suppose that

$$\begin{aligned} (2.3) \quad |a_k| &\leq \frac{2(\beta-1)}{(k-1)(1-\alpha)}(1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}|) \\ &\leq \frac{2(\beta-1)}{(k-1)(1-\alpha)} \prod_{j=1}^{k-2} \left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right) \quad (k \geq 3). \end{aligned}$$

Then we see

$$(2.4) \quad 1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}| \leq \prod_{j=1}^{k-2} \left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right).$$

By using (2.3) and (2.4),

$$\begin{aligned} |a_{k+1}| &\leq \frac{2(\beta-1)}{k(1-\alpha)}(1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}| + |a_k|) \\ &\leq \left(1 + \frac{2(\beta-1)}{(k-1)(1-\alpha)}\right) \frac{2(\beta-1)}{k(1-\alpha)} \prod_{j=1}^{k-2} \left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right) \\ &\leq \frac{2(\beta-1)}{k(1-\alpha)} \prod_{j=1}^{k-1} \left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right). \end{aligned}$$

This completes the proof of the Theorem. □

Corollary 2.2. *If $f(z) \in \mathcal{ND}(\alpha, \beta)$, then*

$$|a_2| \leq \frac{2(\beta-1)}{2(1-\alpha)}$$

and

$$|a_n| \leq \frac{2(\beta-1)}{n(n-1)(1-\alpha)} \prod_{j=1}^{n-2} \left(1 + \frac{2(\beta-1)}{j(1-\alpha)}\right) \quad (n \geq 3).$$

Proof. From $f(z) \in \mathcal{ND}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha, \beta)$, replacing a_n by na_n in Theorem 2.2, we have the corollary. □

3 Applications of Jack's lemma for the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$

In this section, some applications of Jack's lemma for $f(z)$ belonging to the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ are discussed. Next lemma was given by Jack [1].

Lemma 3.1. *Let the function $w(z)$ be analytic in \mathbb{U} with $w(0) = 0$. If*

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)|,$$

then

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \geq 1$.

Theorem 3.1. *If $f(z) \in \mathcal{MD}(\alpha, \beta)$, then*

$$\left| \left(\frac{f(z)}{z} \right)^{\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}} - 1 \right| < 1 + \delta \quad (\delta \geq 0)$$

for some $\alpha(\alpha \leq 0)$ and $\beta(\beta > 1)$, or

$$\left| \left(\frac{f(z)}{z} \right)^{\frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)}} - 1 \right| < 1 + \delta \quad (\delta \geq 0)$$

for some $\alpha(\alpha \leq -1)$ and $\beta(\beta > 1)$.

Proof. Let us define

$$\gamma = \frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)} > 0,$$

for $\alpha \leq 0$ and $\beta > 1$, and

$$\gamma = \frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)} < 0$$

for $\alpha \leq -1$ and $\beta > 1$. Further, let the function $w(z)$ be defined by

$$w(z) = \frac{\left(\frac{f(z)}{z} \right)^\gamma - 1}{1 + \delta} \quad (\delta \geq 0)$$

which is equivalent to

$$\frac{z f'(z)}{f(z)} - 1 = \frac{(1+\delta) z w'(z)}{\gamma \{(1+\delta) w(z) + 1\}}.$$

Then we see that $w(z)$ is analytic in \mathbb{U} , and $w(0) = 0$. On the other hand, if $f(z) \in \mathcal{MD}(\alpha, \beta)$ ($\alpha \leq 0, \beta > 1$), then

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| = 1 + \frac{1}{\gamma} \operatorname{Re} \left(\frac{(1+\delta)zw'(z)}{(1+\delta)w(z)+1} \right) - \frac{\alpha}{|\gamma|} \left| \frac{(1+\delta)zw'(z)}{(1+\delta)w(z)+1} \right| < \beta.$$

Furthermore, if there is a point z_0 ($z_0 \in \mathbb{U}$), which satisfies

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 3.1 gives us that

$$\begin{aligned} & 1 + \frac{1}{\gamma} \operatorname{Re} \left(\frac{(1+\delta)z_0w'(z_0)}{(1+\delta)w(z_0)+1} \right) - \frac{\alpha}{|\gamma|} \left| \frac{(1+\delta)z_0w'(z_0)}{(1+\delta)w(z_0)+1} \right| \\ &= 1 + \frac{k(1+\delta)}{\gamma} \operatorname{Re} \left(\frac{1}{(1+\delta)+e^{-i\theta}} \right) - \frac{\alpha k(1+\delta)}{|\gamma|} \left| \frac{1}{(1+\delta)+e^{-i\theta}} \right| \\ &= 1 + \frac{k(1+\delta)}{\gamma} \cdot \frac{1+\delta+\cos\theta - \alpha\sqrt{(1+\delta)^2+2(1+\delta)\cos\theta+1}}{(1+\delta)^2+2(1+\delta)\cos\theta+1} = F(\theta). \end{aligned}$$

When $\gamma > 0$,

$$\begin{aligned} F(\theta) &\geq 1 + \frac{k(1+\delta)(1-\alpha)}{\gamma(2+\delta)} \\ &\geq 1 + \frac{(1+\delta)(1-\alpha)}{\gamma(2+\delta)} = \beta, \end{aligned}$$

because

$$\gamma = \frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}.$$

Further, when $\gamma < 0$,

$$\begin{aligned} F(\theta) &\geq 1 + \frac{k(1+\delta)(1+\alpha)}{\gamma(2+\delta)} \\ &\geq 1 + \frac{(1+\delta)(1+\alpha)}{\gamma(2+\delta)} = \beta. \end{aligned}$$

This contradicts our condition of the Theorem. Thus there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This completes the proof of the Theorem. \square

Corollary 3.1. *If $f(z) \in \mathcal{ND}(\alpha, \beta)$, then*

$$\left| (f'(z))^{\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}} - 1 \right| < 1 + \delta \quad (\delta \geq 0)$$

for some $\alpha(\alpha \leq 0)$ and $\beta(\beta > 1)$, or

$$\left| (f'(z))^{\frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)}} - 1 \right| < 1 + \delta \quad (\delta \geq 0)$$

for some $\alpha(\alpha \leq -1)$ and $\beta(\beta > 1)$.

Proof. Replacing $f(z)$ by $zf'(z)$ in Theorem 3.1, we have the corollary 3.1. \square

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